Math 246B Lecture 16 Notes

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1 Jensen's Formula

1.1 Example of entire functions of finite order

Last time, we talked about entire holomorphic functions of finite order ($|f(z)| \leq Ce^{|z|^\sigma}$ for some $\sigma \in \mathbb{R}$).

Proposition 1.1. *Let f be entire of finite order ρ which is nonvanishing. Then $f = e^g$, where g is a polynomial of degree ρ.*

Proof. Write $f = e^g$, where g is entire. For any $\varepsilon > 0$, there exists a constant C_ε such that

$$|f(x)| \leq C_\varepsilon e^{|z|^{\rho+\varepsilon}}.$$

So $\operatorname{Re}(g(z)) \leq |z|^{\rho+\varepsilon} + \tilde{C}_\varepsilon$. By the Borel-Carathéodory inequality (proved in homework), g is a polynomial of degree $\leq \rho$. As f has order ρ, we get $\deg(g) = \rho$. $\square$

1.2 Jensen's formula

Theorem 1.1 (Jensen's formula)**.** *Let $f \in \operatorname{Hol}(|z| < R)$, and assume that $f(0) \neq 0$. Let $0 < r < R$, and let $z_1, \dots, z_n$ be the zeros of f in the disc $|z| < r$, each zero repeated according to its multiplicity. Set $r_j = |z_j|$ for each $1 \leq j \leq n$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\varphi})|\, d\varphi = \log\left(\frac{|f(0)|r^n}{r_1 \cdots r_n}\right).$$

If f has no zeros, this integral equals $\log|f(0)|$.

Proof. Replacing $f(z))$ by $f(rz)$, we can assume that $r = 1$. Split into cases of increasing generality:

1. $f \neq 0$ on $|z| \leq 1$: Then $\log|f|$ is harmonic in a neighborhood of $|z| \leq 1$, and Jensen's formula follows from the mean value property.

2. $f \neq 0$ on $|z| = 1$: Let

$$B_j(z) = \frac{\overline{z_j}(z - z_j)}{r_j(\overline{z}_j z - 1)}.$$

This is called a **Blaschke factor**. Then B_j is holomorphic near $|z| \leq 1$. B_j has a simple zero atz_j only, and $|B_j(z)| \leq 1$ when $|z| = 1$. Define $g = f/(B_1 \cdots B_n)$; g is holomorphic near $|z| \leq 1$, nonvanishing, and $|g| = |f|$ when $|z| = 1$. Apply the previous step to g to get

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\varphi})| \, d\varphi = \log |g(0)| = \log \left(\frac{|f(0)|}{r_1 \cdots r_n} \right).$$

3. f has (finitely many) zeros on $|z| = 1$: Apply Jensen's formula to $|z| < r$, where $r < 1$ is close to 1:

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| \, d\varphi = \log |g(0)| = \log \left(\frac{|f(0)| r^n}{r_1 \cdots r_n} \right).$$

Let $r \to 1$, and pass to the limit using dominated convergence. If $f(e^{i\varphi_0}) = 0$, estimate $|\log |f(re^{i\varphi})|$ as $r \to 1$ and $|\varphi - \varphi_0|$ is small: $f(z) = (z - e^{i\varphi_0})^m g(z)$, where g is non-vanishing. We need to consider only $|\log |r - e^{i\psi}||$ as $r \to 1$ and ψ is near 0. We get that $|\log |r - e^{i\psi}|| \leq C(1 + \log(1/|\psi|))$. In particular,

$$|r - e^{i\psi}|^2 = r^2 + 1 - 2r\cos(\psi) = r\psi^2 + O(\psi^4),$$

where we have used $\cos(\psi) = 1 - \psi^2/2 + O(\psi^4)$. Altogether, if $\varphi_1, \cdots \varphi_k$ are the arguments of the zeros of f along the circle $|z| = 1$, we get:

$$|\log |f(re^{i\varphi})| \leq C \left(1 + \sum_{j=1}^{k} \log_+ \left(\frac{1}{|\varphi - \varphi_j|} \right) \right) \in L^1,$$

where $\log_+(t) = \max(\log(t), 0)$. So we can indeed apply the dominated convergence theorem to get Jensen's formula. □

1.3 Number of zeros in a disc

Corollary 1.1. *Let $f \in \mathrm{Hol}(|z| < R)$, and let $n = n(r)$ be the number of zeros of f in $|z| < r$, counted with multiplicities. Let the zeros be $z_1, \ldots, z_{n(r)}$ with $r_j = |z_j|$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| \, d\varphi - \log |f(0)| = \int_0^r \frac{n(t)}{t} \, dt.$$

Proof. Rewrite Jensen's formula using the following computation:

$$\begin{aligned}
\log\left(\frac{r^n}{r_1\cdots r_n}\right) &= \sum_{j=1}^{n}\int_{r_j}^{r}\frac{1}{t}\,dt \\
&= \sum_{j=1}^{n}\int_{0}^{r}\frac{\mathbb{1}_{(r_j,\infty)}(r)}{t}\,dt \\
&= \int_0^r \frac{1}{t}\underbrace{\left(\sum_{j=1}^{n}\mathbb{1}_{(r_j,\infty)}(r)\right)}_{=n(t)}\,dt \\
&= \int_0^r \frac{n(t)}{t}\,dt.
\end{aligned}$$

□

Remark 1.1. In particular,

$$\int_0^r \frac{n(t)}{t}\,dt \geq \int_{r/2}^{r}\frac{n(t)}{t}\,dt \geq n(r/2)\log(2).$$

Next time, we will use Jensen's formula to prove the following fact about entire functions of finite order.

Theorem 1.2. *Let f be entire of finite order ρ, and let $n(r) = |\{z : |z| < r. f(z) = 0\}|$. Then for all $\varepsilon > 0$ and $r \geq 1$,*

$$n(r) \leq C_\varepsilon r^{\rho+\varepsilon}.$$